Investigating Stability Property and Region of Absolute Stability of an Implicit One - step Linear Multiderivative Scheme for Solving Stiff Ordinary Differential Equations.

Olumurewa O. K. E.
Department of Mathematical Sciences,
Kings University, Odeomu, Osun State
ok.olumurewa@kingsuniversity.edu.ng

Abstract

This study investigated the absolute stability property as well as the region of absolute stability of an implicit one – step multiderivative scheme of the form:

$$\sum_{j=0}^{1}\alpha_j\, y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{1}\beta_{ij}\, y^i{}_{n+j}\,,\ \alpha_k = +1,\ i = 1-6 \text{ (} i \text{ is the order of derivative).}$$

For the analysis of these properties, the study verified the root conditions, used Dalhquist stability test equation as well as boundary locus method on the variants of the scheme. The Region of Absolute Stability (RAS) for each variant of the scheme was determined by plotting the graph of absolute stability on the complex plain. The study showed that the third and fourth derivative methods have wider region of absolute stability hence the reason why the methods yielded more accurate results than other variants of the scheme when the methods were used to solve sampled stiff initial value problem of first order ordinary differential equation.

Keywords: *Zero – stability, Absolute stability, Region of absolute stability, Complex plain, Stiff, Ordinary Differential Equation, Multiderivative Method.*

1.0 Introduction

Numerical methods are essentially needed when problems that lead to equations that cannot be solved analytically are encountered. Examples of such problems are solution of large systems of algebraic equations, solution of differential equations and evaluation of integrals. Numerical analysts are therefore concerned with stability which is a concept that refers to the sensitivity of the solution of a problem to small changes in the data or the parameters of the problem.

According to Lambert (2000), the numerical solution of a one-step method depends on the initial condition y_0 while the numerical solution of a k-step method depends on all the k starting values, $y_0, y_1, y_2, \dots, y_{k-1}$. It is thus of interest whether the numerical solution is stable with respect to perturbations in the starting value(s).

According to Suli and Mayer (2003), a linear multistep method is zero-stable for a certain differential equation on a given time interval if a perturbation in the starting values of size ε causes the numerical solution over that time interval to change by no more than $K\varepsilon$ for some value of K which does not depend on the step size h, this is called "zero-stability" because it is enough to check the condition for the differential equation $y^1 = 0$. If the roots of the characteristic polynomial ρ all have modulus less than or equal to 1 and the roots of modulus 1 are of multiplicity 1, then the root condition is satisfied. A linear multistep method is therefore zero-stable if and only if the root condition is satisfied, that is, if the roots of the generating polynomial $\rho(r)$ satisfy the following conditions

(i) all roots r_i satisfy $|r_i| \leq 1$ (ii) all roots r_i with $|r_i| = 1$ are simple.

If either of these two conditions is violated, the method is unstable. This is equivalent to $\rho(r)$ having either a root r_i outside the unit circle or a multiple root on the circle.
According to Ezzeddine and Hojjati (2012), the form of stability needed is sometimes stronger than zero stability, error is well behaved for the particular time step being used, although, the exponential growth of errors does not contradict zero stability or convergence of the method in any way. To determine whether a numerical method will produce reasonable result with a given value of $k > 0$, a notion of stability that is different from zero stability is needed. The one which is most basic is absolute stability whose notion is based on the linear test equation: $y^1 = \lambda y$.
Stiff initial value problems are frequent occurrences in the mathematical formulation of physical situations in control theory and mass kinetics where processes with widely varying time space are usually considered. In some chemical engineering problems, dynamic balance must be solved with accumulation that terms differ by several orders of magnitude. This corresponds to physical processes where some dependent variables relax very fast while others approach the stationary state slowly. This type of problems is called "stiff" and are difficult to solve. Stiff problems often arise in reactor engineering (radical reactions, complex reactions with some of them very fast) and in system engineering (dynamic regime of a distillation column with a mixture containing one very volatile component or one component with very low concentration).
Problems in dynamics of counter-current separation devices or systems of interacting devices lead to systems of hundreds of ordinary differential equations. Solution of such problems often requires special algorithms.
In recent years, numerous works have focused on the development of more advanced and efficient methods for stiff problems. A potentially good numerical method for the solution of stiff systems of ordinary differential equations must have good accuracy and some reasonable wide region of absolute stability. Backward differentiation formulas (BDF) are also implicit methods which are especially used for the solution of stiff differential equations because they are absolutely – stable methods. [Ebadi and Gokhale (2010)]
According to Lei (2008), a linear multistep method is called absolutely – stable if the region of its stability extends into the left half of the complex plain and if errors introduced in one - time step do not grow in future time step. In this study, the linear multiderivative scheme developed in Famurewa (2011) will be investigated to ascertain the capability of the methods to solving stiff ordinary differential equations.

2.0 REASERCH METHODOLOGY

Setting i = 1,2,3,4,5 and 6 in the one – step multiderivative scheme of the form:

$$\sum_{j=0}^{1} \alpha_j \, y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{1} \beta_{ij} \, y^i{}_{n+j}, \, a_1 = 1, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

yielded one step first derivative method of the form:

$$y_{n+1} = y_n + \frac{h}{2}(y_{n+1}^1 + y_n^1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

one step second derivative method of the form:

$$y_{n+1} = y_n + \frac{h}{2}(y_{n+1}^1 + y_n^1) - \frac{h^2}{12}(y_{n+1}^{11} + y_n^{11}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

one step third derivative method of the form:

$$y_{n+1} = y_n + \frac{h}{2}(y_{n+1}^1 + y_n^1) - \frac{h^2}{10}(y_{n+1}^{11} - y_n^{11}) + \frac{h^3}{120}(y_{n+1}^{111} + y_n^{111}) \qquad \ldots\ldots\ldots\ldots (4)$$

one step fourth derivative method of the form:

$$y_{n+1} = y_n + 0.5h(y_{n+1}^1 + y_n^1) - 0.11h^2(y_{n+1}^{11} - y_n^{11}) + 0.01h^3(y_{n+1}^{111} + y_n^{111}) \quad \ldots. (5)$$

one step fifth derivative method of the form:

$y_{n+1} = y_n + 0.5h(y_{n+1}^1 + y_n^1) - 0.11h^2(y_{n+1}^{11} - y_n^{11}) + 0.01h^3(y_{n+1}^{111} + y_n^{111})$ …….. (6)

and one step sixth derivative method of the form:

$y_{n+1} = y_n + 0.5h(y_{n+1}^1 + y_n^1) - 0.11h^2(y_{n+1}^{11} - y_n^{11}) + 0.02h^3(y_{n+1}^{111} + y_n^{111})$ …….. (7)

as reported by Famurewa and Olorunsola (2013)

2.1 Analysis of the Zero-stability property of the method

According to Babatola *et al.;* (2007), a L.M.M of the form:

$y_{n+k} = \alpha_0 y_n + h(\beta_1 {y^1}_{n+k} + \beta_0 {y^1}_n)$

with first characteristic polynomial

$\rho(r) = r^{n+k} - \alpha_0 r^n$

is said to be zero-stable if the root of the first characteristic polynomial $\rho(r)$ has modulus less than or equal to 1; $(|r| \leq 1)$

Equations (2) – (7) have their first characteristic polynomial $\rho(r)$ as:

$$r^{n+1} - r^n = 0$$
$$r^n(r-1) = 0$$

and the roots are $r = 0$ *or* $r = 1$ [roots are within a unit circle]

Hence, the methods are all zero stable as reported in Famurewa and Olorunsola (2013)

2.2 Analysis of Absolute stability

A linear multistep method is absolutely stable if the region of its stability extends into the left half of the complex plane, Butcher (2008).

Boundary locus method and Dalhquist stability test equation $y^1 = \lambda y$ were adopted in equations (2) – (7).

2.2.1 For the one step first derivative method of equation (2):

Setting $z = \lambda h$ and simplifying gives

$$\left|\frac{1-\frac{z}{2}}{1+\frac{z}{2}}\right| < 1$$

that is $$-1 < \frac{1-\frac{z}{2}}{1+\frac{z}{2}} < 1$$

Simplifying further gives $z < 2$ and $z < 0$.

Since $|u(z)| < 1$, then the method is A - stable

2.2.2 For the one step second derivative method of equation (3):

$$\left|\frac{1-\frac{z}{2}+\frac{z^2}{12}}{1+\frac{z}{2}+\frac{z^2}{12}}\right| < 1$$

Simplifying gives: $z^2 > -12$ and $z < 0$ considering the real part gives $|u(z)| < 0$, hence the method is A - stable

2.2.3 For the one step third derivative method of equation (4):

$$\left|\frac{1+\frac{z}{2}+\frac{z^2}{10}+\frac{z^3}{120}}{1-\frac{z}{2}+\frac{z^2}{10}-\frac{z^3}{120}}\right| < 1$$

Simplifying gives: $z^2 > -60$ and $z < 0$ considering the real part gives $|u(z)| < 0$, hence the method is A - stable

2.2.4 For the one step fourth derivative method of equation (5):

$$\left|\frac{1 + 0.5z + 0.11z^2 + 0.01z^3}{1 - 0.5z + 0.11z^2 - 0.01z^3}\right| < 1$$

Simplifying gives: $z^2 > -9.09$ and $z < 0$ considering the real part gives $|u(z)| < 0$, hence the method is A - stable

2.2.5 For the one step fifth derivative method of equation (6):

$$\left|\frac{1 + 0.5z - 0.11z^2 + 0.01z^3}{1 - 0.5z + 0.11z^2 - 0.01z^3}\right| < 1$$

Simplifying considering the real part gives $|u(z)| < 0$, hence the method is A - stable

2.2.6 For the one step sixth derivative method of equation (7):

$$\left|\frac{1 + 0.5z + 0.11z^2 + 0.02z^3}{1 - 0.5z + 0.11z^2 - 0.02z^3}\right| < 1$$

Simplifying considering the real part gives $|u(z)| < 0$, hence the method is A - stable

3.0 Determination of Region of Absolute Stability

To determine the region of absolute stability of the methods, the solutions to $|u(z)| = e^{i\theta}$ is plotted at an interval of 30^0 between $0^0 \leq \vartheta \leq 180^0$. This gives the interval of absolute stability as $(0,\infty)$ in figures I – 6.

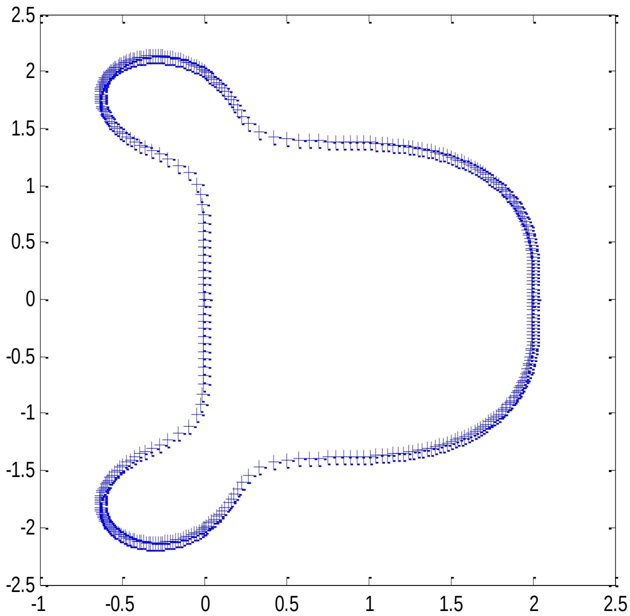

Figure 1: Region of A - stability for one - step first derivative scheme.

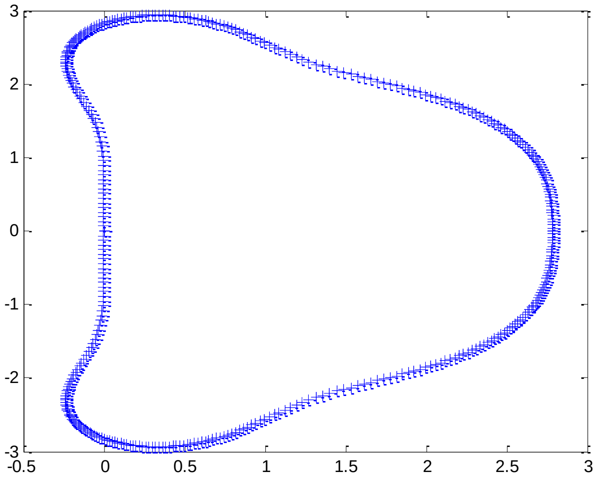

Figure 2: Region of A - stability for one - step second derivative scheme.

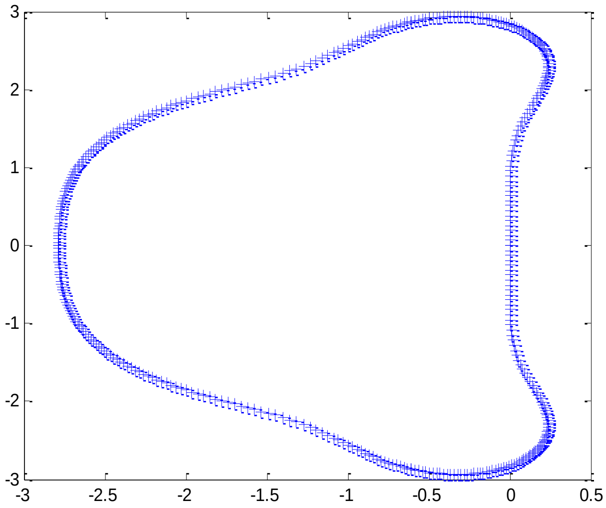

Figure 3: Region of A - stability for one - step third derivative scheme.

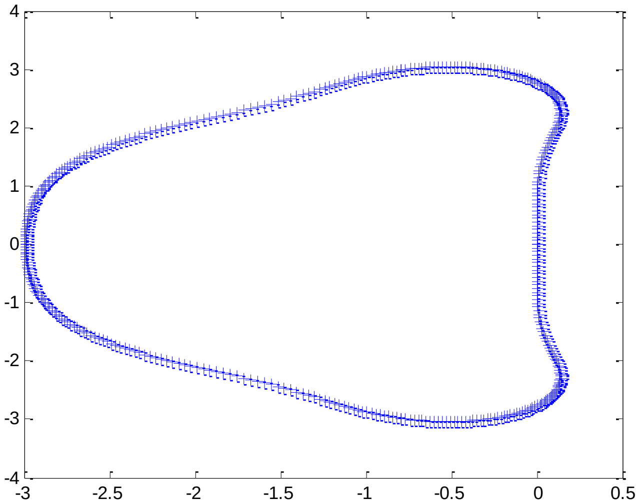

Figure 4: Region of A - stability for one - step fifth derivative scheme.

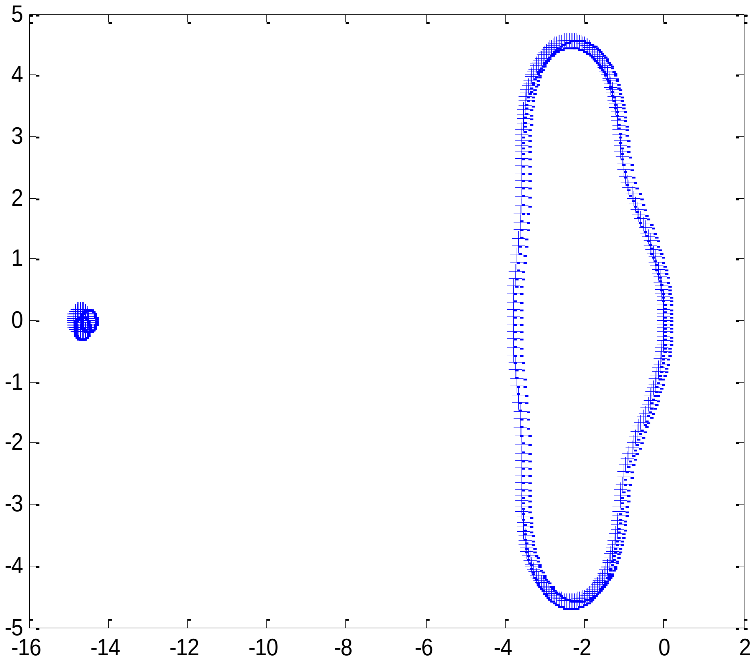

Figure 5: Region of A - stability for one - step fifth derivative scheme.

.

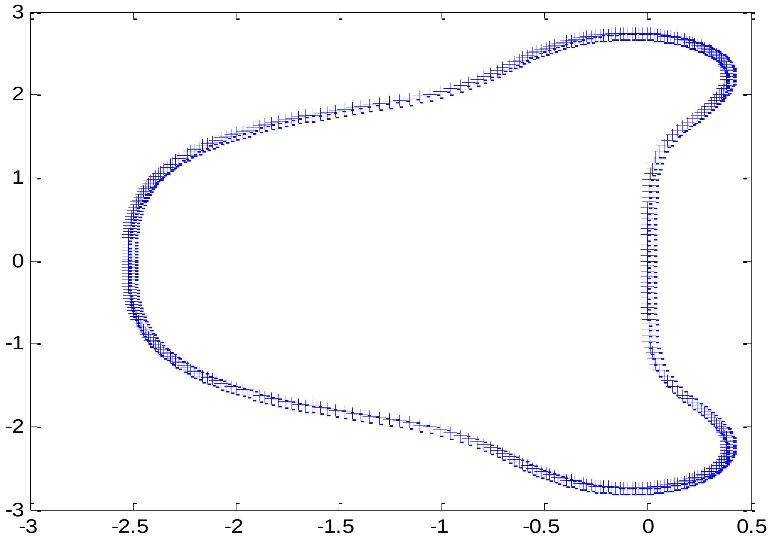

Figure 6: Region of A - stability for one - step sixth derivative scheme

4.0 Conclusion

The study showed that the implicit one step multiderivative methods are absolutely stable hence are capable of solving stiff initial value problems of first order ordinary differential equations. Figures 1 – 6 showed that the one step third derivative method of equation (4) and one step fourth derivative method of equation (5) covered a wider part of the left half of the complex plain than the other methods examined. This establishes the reason for the better results obtained when the one - step third derivative method and one – step fourth derivative method were used to solve a stiff first order ordinary differential equation as reported in Famurewa and Olorunisola (2013).

References

Babatola P.O., Ademiluyi R.A. and Areo E.A. (2008), K-Step Rational Runge-Kutta Method for Solution of Stiff System of Ordinary Differential Equations. *Journal of Mathematics and Statistics* 4 (3): 130-137.

Butcher, J. (2008). Numerical Methods for Ordinary Differential Equations (2nd ed.). *John Wiley and Sons Ltd, West Sussex*

Ebadi M. and Gokhale M.Y. (2010). Hybrid BDF methods for the numerical solutions of ordinary differential equations. *Numer. Algor.*, 55:1-17.

Ezzeddine A.K and Hojjati G (2012). Third derivative multistep methods for stiff systems. *International Journal of Non linear Sciences.* 14(4):443 - 450.

Famurewa O. K. E. (2011). Improving the Accuracy of the Conventional Implicit Linear Multistep Methods for Integration of Stiff Ordinary Differential Equations .*International journal of Natural& Applied Sciences.*3(2): 140-150

Famurewa, O.K.E. and Olorunsola, S.A. (2013). The effect of involving more derivative properties on the accuracy of an implicit one – step method. *Asian Journal of Natural & Applied Sciences*, 2(3): 142-157.

Lambert J.D (2000). Numerical method for ordinary differential systems of initial value problems. *John Wiley and Sons*, New York.

Lei Liu, (2008). Linear multistep methods. University of Technology, Eindhoven. Multistep methods_ Accessed on 11/09/2014.

Süli, Endre and Mayers, David (2003), *An Introduction to Numerical Analysis*, Cambrige University Press, ISBN 0-521 -00794-1.